

PEGASIS

Practical Efficient Class Group Action using 4-dimensional isogenies

Joint with Pierrick Dartois, Jonathan Komada Eriksen, Tako Boris Fouotsa, Arthur Herledan Le Merdy, Riccardo Invernizzi, Damien Robert, Frederik Vercauteren and Benjamin Wesolowski

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Ryan Rueger

IBM Research Zurich & Technical University of Munich

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Cryptographic Group Actions

Easy to compute gx for all $g \in G$ and $x \in X$, but hard to recover g from (x, gx)

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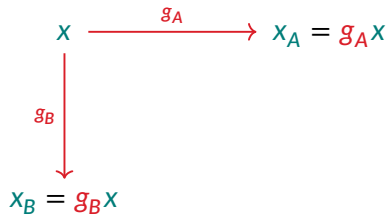
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Example Commutative group gives Non-Interactive Key-Exchange

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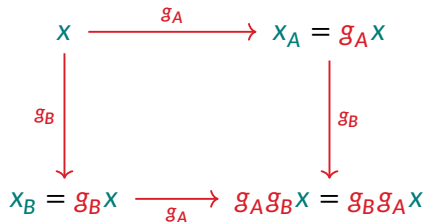
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Quantum resistance

Best known generic (quantum) algorithm to invert the action is subexponential

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Isogeny class-group action only known commutative, quantum resistant
cryptographic group action

Abelian Varieties & Isogenies

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Abelian Variety (Formal)

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Group scheme over a field

Abelian Varieties & Isogenies

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Group scheme over a field with one property from each set

- (i) projective, proper
- (ii) geometrically irreducible, irreducible, geometrically connected, connected
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First isomorphism theorem

$$\{G \subseteq A \text{ finite}\} \leftrightarrow \{\text{separable isogenies } A \rightarrow *\} / \{\text{iso}\}$$

$$G \mapsto \varphi_G \text{ with } \ker(\varphi_G) = G \quad (\deg(\varphi_G) = \#G)$$

$$\ker(\varphi) \leftarrow \varphi$$

The isogeny class-group action

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Theorem

The action on oriented supersingular elliptic curves is free and has at most 2 orbits

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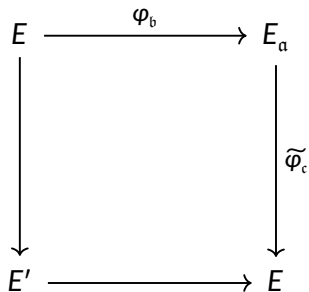
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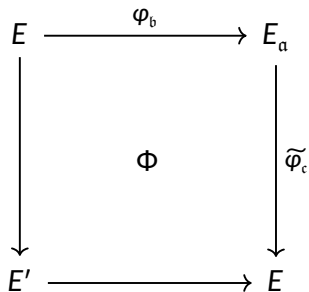


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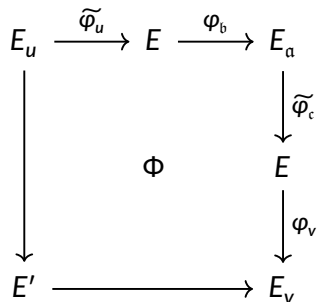
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Solvability of the norm equation: Conflict

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Core Problem

Even the smallest equivalent ideals are too big...but not by much

Solvability of the norm equation: An idea

Recall

Ideals of $\mathcal{O}$ prime to the conductor factorise uniquely as product of prime ideals

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Idea

Factor the ideals $\mathfrak{b} = \mathfrak{b}_e \mathfrak{b}_k, \mathfrak{c} = \mathfrak{c}_e \mathfrak{c}_k$ with action of $\mathfrak{b}_e, \mathfrak{c}_e$ “easy”

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Ensure $N(\mathfrak{b}_e), N(\mathfrak{c}_e)$ are products of small primes split in $\mathcal{O}$

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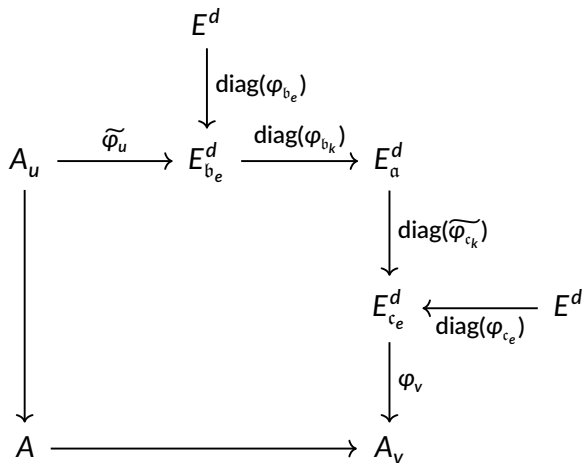
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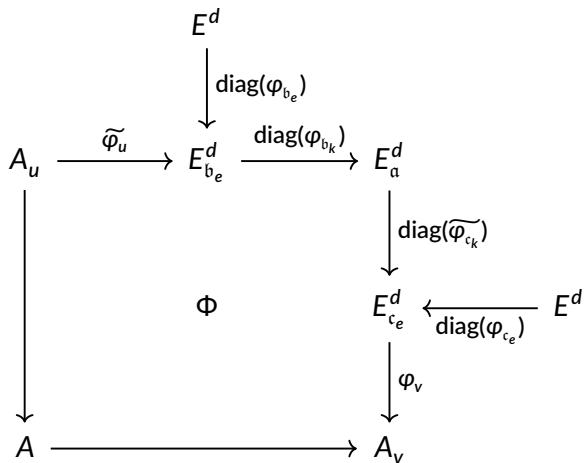
Concretely $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Compute $[\mathfrak{b}_e] \cdot E$ with successive Elkies isogenies

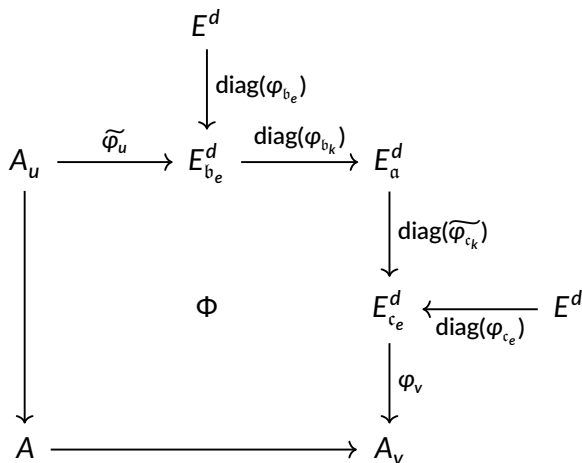
Solvability of the norm equation: The diagram



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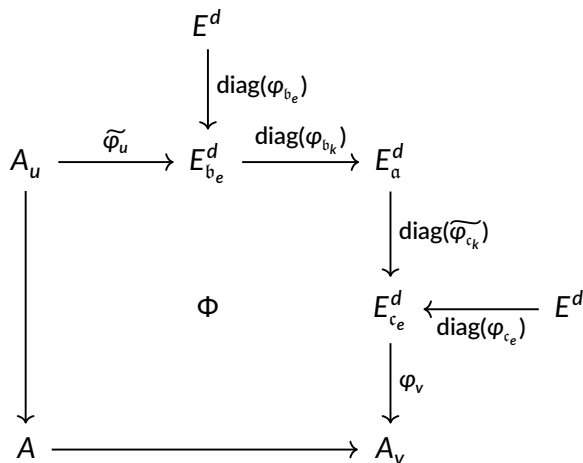


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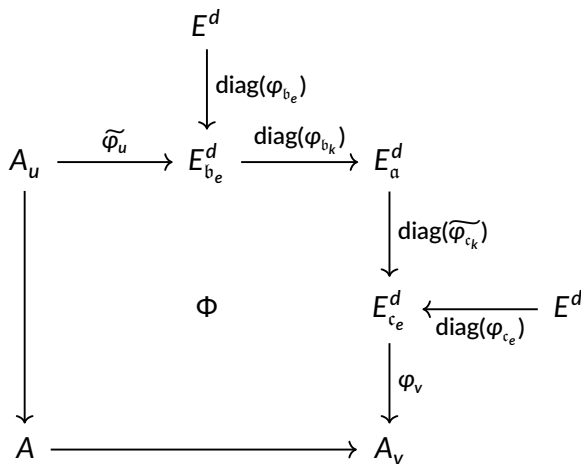
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Norm equation $\deg_p(\Phi) = uN(b_k) + vN(c_k) \stackrel{!}{=} 2^f$

$$\ker(\Phi) = \left\{ (uN(b_e)x, \varphi_v \text{diag}(\widetilde{\varphi_{c_k}} \varphi_{b_k}) \widetilde{\varphi}_u(x)) \mid x \in A_u[2^f] \right\} \quad \text{and} \quad \widetilde{\varphi_{c_k}} \varphi_{b_k} = \frac{1}{N(b_e)N(c_e)} \varphi_{c_e} \varphi_{\bar{c}b} \widetilde{\varphi_{b_e}}$$

Solvability of the norm equation: The diagram



$$\varphi_{\bar{c}b} = \widetilde{\varphi}_c \varphi_b = \widetilde{\varphi}_{c_e} \widetilde{\varphi}_{c_k} \varphi_{b_k} \varphi_{b_e}$$

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Constructing isogenies of prescribed degree

The isogenies φ_u, φ_v

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Dimension 1 $\rightsquigarrow$ Dimension 2 Kani-isogeny Φ

Requires knowledge of the endomorphism ring (SQISign2D)

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Sums of squares

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Sums of squares

Dimension 4 $\rightsquigarrow$ Dimension 8 Kani-isogeny Φ

Zahrin's trick

An algorithm for solving the norm equation

Input: $\alpha \in \mathcal{O}$, $p = c2^e - 1$, Set of small split primes $\mathfrak{B}$

Output: Return $b_e, b_k, c_e, c_k, \varphi_u, \varphi_v$ such that $uN(b_k) + vN(c_k) = 2^f \leq 2^{e-3}$

An algorithm for solving the norm equation

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6. Return $\mathfrak{b}_e, \mathfrak{b}_k, \varphi_u, \varphi_v$

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	37.137	24.506	0.968	542.416
Rerandomisations:	0.499	0	0	11
$\log(2^f = uN(b_k) + vN(c_k))$	4076.080	4077	4058	4081
UV solutions tried:	38952.563	19701	10	548118
3-Elkies steps for $\varphi_{b_e}, \varphi_{c_e}$:	4.950	4	0	17
7-Elkies steps for $\varphi_{b_e}, \varphi_{c_e}$:	2.658	2	0	9
11-Elkies steps for $\varphi_{b_e}, \varphi_{c_e}$:	2.010	2	0	9
17-Elkies steps for $\varphi_{b_e}, \varphi_{c_e}$:	1.691	1	0	7
19-Elkies steps for $\varphi_{b_e}, \varphi_{c_e}$:	1.567	1	0	6
3-Elkies steps for φ_u, φ_v :	0.496	0	0	1
7-Elkies steps for φ_u, φ_v :	0.259	0	0	1
11-Elkies steps for φ_u, φ_v :	0.160	0	0	1
17-Elkies steps for φ_u, φ_v :	0.000	0	0	0
19-Elkies steps for φ_u, φ_v :	0.106	0	0	1

Table: Times, rerandomisations and elkies steps required for $p = 63 \cdot 2^{4084} - 1$.

Implementation Results

This all works!

		Lang.	500	1000	1500	2000	4000
Restricted	CSIDH*	C	40ms				
	SQALE*	C					5.75s**
	dCTIDH*	C				350ms**	
Unrestricted	SCALLOP*	C++	35s	750s			
	SCALLOP-HD*	Sage	88s	1140s			
	PEARL-SCALLOP*	C++	30s	58s	710s		
	KLaPoTi	Sage	207s				
		Rust	1.95s				
	PEGASIS	Sage	1.53s	4.21s	10.5s	21.3s	121s

Table: *Measured on different hardware, **Converted from cycles to time @4GHz.

Thank you for your attention

PEGASIS Paper <https://eprint.iacr.org/2025/401> (To appear at Crypto'25)

PEGASIS Implementation <https://github.com/pegasis4d>

Slides <https://rueg.re/siam25>

Ask me anything (I have bonus slides!)

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PPAVs vs Elliptic Curves

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E, E' Elliptic curves

$\varphi : E \rightarrow E'$ isogeny

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...it has *polarised degree* $\deg_p(f) = d$

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A *Principally Polarised Abelian Variety* is an Abelian variety A together with an isomorphism $\lambda_A : A \xrightarrow{\sim} A^\vee$ (which is induced by an ample divisor)

PPAVs vs Elliptic Curves

A, B *Principally Polarised Abelian Varieties*

$f : A \rightarrow B$ *isogeny*

$\tilde{f} = \lambda_B^{-1} f^\vee \lambda_A : B \rightarrow A$ *polarised dual*

f is *polarised isogeny* if $\tilde{f}f = [d]_A$

...it has *polarised degree* $\deg_p(f) = d$

E, E' Elliptic curves

$\varphi : E \rightarrow E'$ isogeny

$\hat{\varphi} = \tilde{\varphi} : E' \rightarrow E$ dual

(They all are)

$\deg_p(\varphi) = \deg(\varphi)$

Example

$\text{diag}_d(f) : A^d \rightarrow B^d$ is polarised (if f is) and $\deg_p(\text{diag}_d(f)) = \deg_p(f)$

Example

$\Phi = (x, y; -y, x)$ in $\text{Mat}_{2 \times 2}(\mathbb{Z})$ is polarised on A^2 with $\deg_p(\Phi) = x^2 + y^2$

Kani's Lemma

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is a polarised isogeny



$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_{11}} & B_1 \\ \downarrow -\varphi_{12} & & \downarrow \widetilde{\varphi}_{21} \\ B_2 & \xrightarrow{\widetilde{\varphi}_{22}} & A_2 \end{array}$$

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Then $\deg_p(\Phi) = \deg_p(\varphi_{11}) + \deg_p(\varphi_{21})$

If additionally $\deg_p(\varphi_{11}), \deg_p(\varphi_{21})$ coprime and $\text{char}(k) \nmid \deg_p(\Phi)$ then

$$\ker(\Phi) = \left\{ \left(\deg_p(\varphi_{11})x, \widetilde{\varphi}_{21}\varphi_{11}(x) \right) \mid x \in A_1[\deg_p(\Phi)] \right\} \subseteq A_1 \times A_2$$

An Algorithm for constructing isogenies of prescribed degree in dimension 2

Input: Integer u , Bound B , Set of small split primes $\mathcal{S}$

Output: Isogeny φ_u with degree u

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6. Return

$$\varphi_u = \begin{pmatrix} \psi_u & 0 \\ 0 & \psi_u \end{pmatrix} \begin{pmatrix} x_u & y_u \\ -y_u & x_u \end{pmatrix}$$

Working over F_p : Fast basis sampling

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Lemma

Let $E : y^2 = g(x)$

An element x_p in F_p lifts to $P = (x_p, y_p)$

- (i) on E with $\text{ord}(P) = 2^{e-1}$ iff $x(p) - x(T_{\text{desc},1})$ a non-zero non-square
(and $g(x_p)$ non-zero square)
- (ii) on E^t with $\text{ord}(P) = 2^{e-1}$ iff $x(p) - x(T_{\text{desc},2})$ non-zero square
(and $g(x_p)$ a non-zero non-square)

Constructing isogenies of prescribed degree

In dimension 2 with QFESTA splitting via 4-dimensional isogeny

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Consider $\gamma_1 = x_1 + y_1\sqrt{\Delta}$ and $\gamma_2 = x_2 + y_2\sqrt{\Delta}$ in $\mathcal{O} = \mathbb{Z} + \sqrt{\Delta}\mathbb{Z}$

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Idea If $N \gg |\Delta|$ then $m = N - |\Delta|(y_1^2 + y_2^2) \geq 0$ often enough that we find $m = x_1^2 + x_2^2$ as sum of squares

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Heuristic If $N \gg |\Delta|$, we can find endomorphism of E^2 with polarised degree N

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Then the 4-dimensional isogeny

$$\Gamma = \begin{pmatrix} \mu_1 & \widetilde{\mu}_2 \\ -\nu_1 & \widetilde{\nu}_2 \end{pmatrix}$$

has polarised degree $N = N_1 + N_2$ and kernel $\{(\deg_p(\mu_1)x, \gamma(x)) \mid x \in E^2[N]\}$

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Our $u \approx \sqrt{\Delta} = \sqrt{p}$

Set $N = u(2^{e/2} - u) \gg |\Delta| = p$ to get 4-dimensional isogeny Γ of degree $2^{e/2}$

Obtain u -isogeny $\mu_i : E^2 \rightarrow A_u$ as component of Γ