Swissogeny Day 2, 20th of March 2025

PEGASIS
Practical Efficient Class Group Action using 4-dimensional isogenies

Joint with Pierrick Dartois, Jonathan Komada Eriksen, Tako Boris Fouotsa, Arthur Herledan Le Merdy, Riccardo Invernizzi, Damien Robert, Frederik Vercauteren and Benjamin Wesolowski

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Results

	Lang.	500	1000	1500	2000	4000
SCALLOP*	C++	35s	750s			
SCALLOP-HD*	Sage	88s	1140s			
PEARL-SCALLOP*	C++	30s	58s	710s		
KLaPoTi	Sage	207s				
	Rust	1.95s				
PEGASIS	Sage	1.53s	4.21s	10.5s	21.3s	121s

Table: Time measured in wall-clock time. Stars indicate different measuring hardware.

Higher dimensional isogenies

A, B *Principally Polarised Abelian Varieties* $\quad E, E'$ Elliptic curves

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Important fact

$f : A \to B$ a polarised isogeny

$$\mathrm{diag}_d(f) = \begin{pmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{pmatrix} : A^d \to B^d$$

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$\deg_{\mathrm{p}}(\mathrm{diag}_d(f)) = \deg_{\mathrm{p}}(f)$

Kani's Lemma

A_i, B_j PPAVs over k, $\varphi_{ij} : A_i \to B_j$ polarised isogenies, $\mathrm{char}(k) \nmid \deg_\mathrm{p}(\varphi_{ij})$

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Then $\deg_p(\Phi) = \deg_p(\varphi_{11}) + \deg_p(\varphi_{21})$

If additionally $\deg_p(\varphi_{11}), \deg_p(\varphi_{21})$ coprime and $\mathrm{char}(k) \nmid \deg_p(\Phi)$ then

$$\ker(\Phi) = \left\{ \left(\deg_p(\varphi_{11})x, \widetilde{\varphi_{21}}\varphi_{11}(x) \right) \mid x \in A_1[\deg_p(\Phi)] \right\} \subseteq A_1 \times A_2$$

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Let (d_1, d_2) coprime positive integers such that $\deg_p(f) = d_1 d_2$

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There exists a PPAV C and isogenies $f_1: A \to C$, $f_2: C \to B$ such that

1. $\deg_p(f_1) = d_1$
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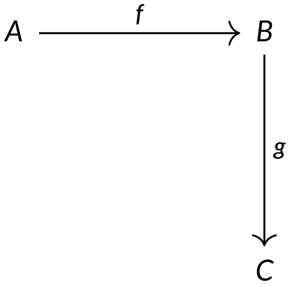
Corollary

$f : A \to B$, $g : B \to C$ polarised isogenies with $\deg_p(f), \deg_p(g)$ coprime



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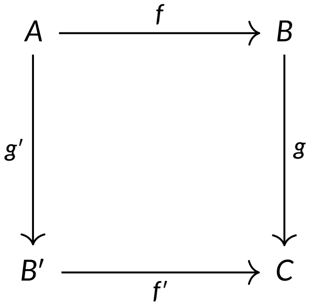

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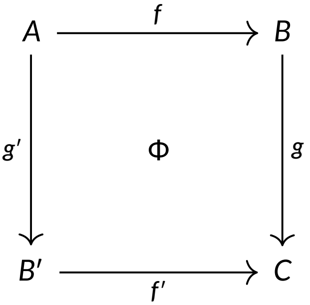

$\rightsquigarrow$

$$\Phi = \begin{pmatrix} f & \tilde{g} \\ -g' & \tilde{f'} \end{pmatrix} : A \times C \to B \times B'$$

The Ideal2Isogeny Construction

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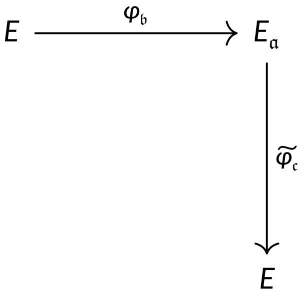

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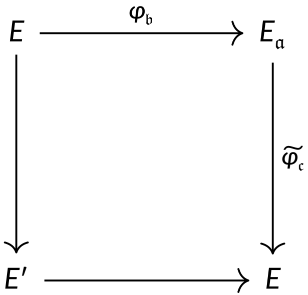

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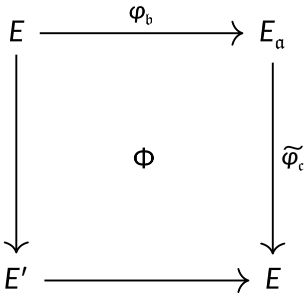

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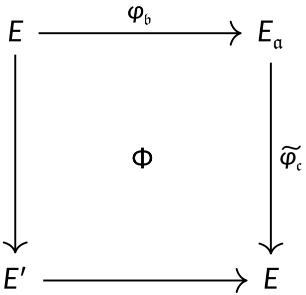

$$\begin{aligned}\ker(\Phi) &= \left\{\left(\mathrm{N}(\mathfrak{b})x, \widetilde{\varphi}_{\mathfrak{c}}\varphi_{\mathfrak{b}}(x)\right) \mid x \in E[\deg_{\mathrm{p}}(\Phi)]\right\} \\ &= \left\{\left(\mathrm{N}(\mathfrak{b})x, \varphi_{\bar{\mathfrak{c}}\mathfrak{b}}(x)\right) \mid x \in E[\deg_{\mathrm{p}}(\Phi)]\right\}\end{aligned}$$

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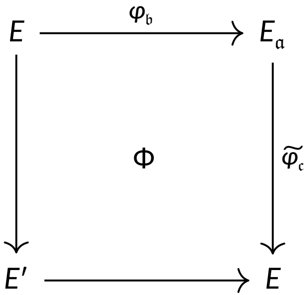

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Norm equation $\deg_{\mathrm{p}}(\Phi) = \mathrm{N}(\mathfrak{b}) + \mathrm{N}(\mathfrak{c}) \overset{!}{=} 2^f$

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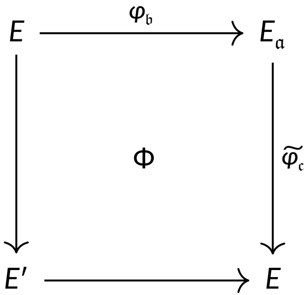

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Requirements

1. $f \leq v_2(p+1) - 3$
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$$\begin{array}{ccccc} E_u & \xrightarrow{\widetilde{\varphi_u}} & E & \xrightarrow{\varphi_{\mathfrak{b}}} & E_{\mathfrak{a}} \\ & & & & \downarrow \widetilde{\varphi_{\mathfrak{c}}} \\ & & & & E \\ & & & & \downarrow \varphi_v \\ & & & & E_v \end{array}$$

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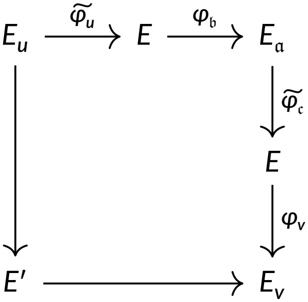

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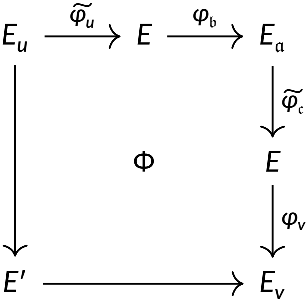

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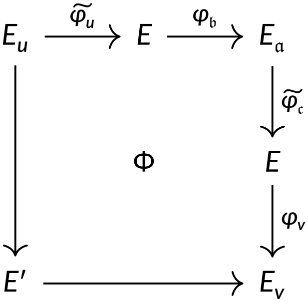

$$\ker(\Phi)$$

$$= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \widetilde{\varphi}_{\mathfrak{c}} \varphi_{\mathfrak{b}} \widetilde{\varphi}_u(x) \right) \mid x \in E_u[\deg_p(\Phi)] \right\}$$

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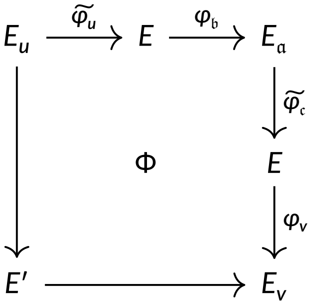

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Norm equation $\deg_p(\Phi) = u\mathrm{N}(\mathfrak{b}) + v\mathrm{N}(\mathfrak{c}) \overset{!}{=} 2^f$

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$$\begin{array}{ccccc} E_u & \xrightarrow{\widetilde{\varphi_u}} & E & \xrightarrow{\varphi_{\mathfrak{b}}} & E_{\mathfrak{a}} \\ \big\downarrow & & & & \big\downarrow \widetilde{\varphi_{\mathfrak{c}}} \\ & & \Phi & & E \\ \big\downarrow & & & & \big\downarrow \varphi_v \\ E' & & \longrightarrow & & E_v \end{array}$$

$$\begin{aligned} &\ker(\Phi) \\ &= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \widetilde{\varphi_{\mathfrak{c}}} \varphi_{\mathfrak{b}} \widetilde{\varphi_u}(x) \right) \mid x \in E_u[\deg_{\mathrm{p}}(\Phi)] \right\} \\ &= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \varphi_{\bar{\mathfrak{c}}\mathfrak{b}} \widetilde{\varphi_u}(x) \right) \mid x \in E_u[\deg_{\mathrm{p}}(\Phi)] \right\} \end{aligned}$$

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1. $f \leq v_2(p+1) - 3$ 2. $u\mathrm{N}(\mathfrak{b}), v\mathrm{N}(\mathfrak{c})$ coprime 3. $u = \deg_{\mathrm{p}}(\varphi_u), v = \deg_{\mathrm{p}}(\varphi_v)$

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$$
\begin{array}{ccccc}
A_u & \xrightarrow{\widetilde{\varphi_u}} & E^d & \xrightarrow{\mathrm{diag}(\varphi_{\mathfrak{b}})} & E_{\mathfrak{a}}^d \\
 & & & & \big\downarrow \mathrm{diag}(\widetilde{\varphi_{\mathfrak{c}}}) \\
 & & & & E^d \\
 & & & & \big\downarrow \varphi_v \\
 & & & & A_v
\end{array}
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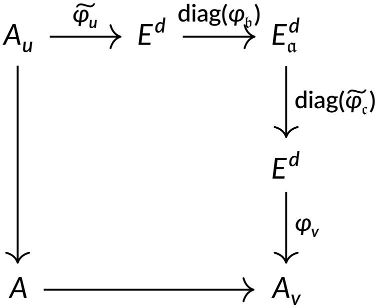

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$$
\begin{array}{ccccc}
A_u & \xrightarrow{\widetilde{\varphi}_u} & E^d & \xrightarrow{\mathrm{diag}(\varphi_{\mathfrak{b}})} & E_{\mathfrak{a}}^d \\
\downarrow & & & & \downarrow \mathrm{diag}(\widetilde{\varphi}_{\mathfrak{c}}) \\
 & & \Phi & & E^d \\
\downarrow & & & & \downarrow \varphi_v \\
A & & \longrightarrow & & A_v
\end{array}
$$

$$
\begin{aligned}
&\ker(\Phi) \\
&= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \mathrm{diag}(\widetilde{\varphi}_{\mathfrak{c}} \varphi_{\mathfrak{b}}) \widetilde{\varphi}_u(x) \right) \mid x \in A_u[\deg_{\mathrm{p}}(\Phi)] \right\} \\
&= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \mathrm{diag}(\varphi_{\bar{\mathfrak{c}}\mathfrak{b}}) \widetilde{\varphi}_u(x) \right) \mid x \in A_u[\deg_{\mathrm{p}}(\Phi)] \right\}
\end{aligned}
$$

The Ideal2Isogeny Construction: Iteration #3

Want to compute $[\mathfrak{a}] \cdot E = E_{\mathfrak{a}}$

Let $[\mathfrak{a}] = [\mathfrak{b}] = [\mathfrak{c}]$, $\varphi_u : E^d \to A_u$, $\varphi_v : E^d \to A_v$

Assume $u\mathrm{N}(\mathfrak{b}), v\mathrm{N}(\mathfrak{c})$ coprime

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Norm equation $\deg_{\mathrm{p}}(\Phi) = u\mathrm{N}(\mathfrak{b}) + v\mathrm{N}(\mathfrak{c}) \overset{!}{=} 2^f$

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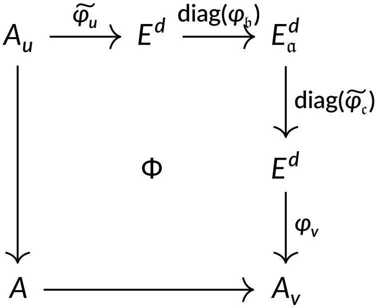

$$\begin{aligned} &\ker(\Phi) \\ &= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \mathrm{diag}(\widetilde{\varphi}_{\mathfrak{c}}\varphi_{\mathfrak{b}})\widetilde{\varphi}_u(x)\right) \mid x \in A_u[\deg_{\mathrm{p}}(\Phi)] \right\} \\ &= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \mathrm{diag}(\varphi_{\bar{\mathfrak{c}}\mathfrak{b}})\widetilde{\varphi}_u(x)\right) \mid x \in A_u[\deg_{\mathrm{p}}(\Phi)] \right\} \end{aligned}$$

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Requirements

1. $f \leq v_2(p+1) - 3$
2. $u\mathrm{N}(\mathfrak{b}), v\mathrm{N}(\mathfrak{c})$ coprime
3. $u = \deg_{\mathrm{p}}(\varphi_u), v = \deg_{\mathrm{p}}(\varphi_v)$

Solvability of the norm equation

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Frobenius Coin Problem

$u\mathrm{N}(\mathfrak{b}) + v\mathrm{N}(\mathfrak{c}) = 2^f$ has a solution if $\mathrm{N}(\mathfrak{b})\mathrm{N}(\mathfrak{c}) \leq 2^f - \mathrm{N}(\mathfrak{b}) - \mathrm{N}(\mathfrak{c})$

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Minkowski Bound

Every class in $\mathrm{Cl}(\mathcal{O})$ has a representative of norm at most $\sqrt{\mathrm{Disc}(\mathcal{O})}$

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Every class in $\mathrm{Cl}(\mathbb{Z}[(1+\sqrt{-p})/2])$ has two representatives $\mathfrak{b}, \mathfrak{c}$, with $\mathfrak{b} \neq \lambda\mathfrak{c}$, such that $p \leq \mathrm{N}(\mathfrak{b})\mathrm{N}(\mathfrak{c}) \leq 2p$

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We only permit $f < e$, but $\mathrm{N}(\mathfrak{b})\mathrm{N}(\mathfrak{c}) \approx p = c2^e - 1 \not\lesssim 2^f - \mathrm{N}(\mathfrak{b}) - \mathrm{N}(\mathfrak{c})$

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Conclusion The norms $\mathrm{N}(\mathfrak{b}), \mathrm{N}(\mathfrak{c})$ are too big!

Solvability of the norm equation: An idea

Recall

Ideals of $\mathcal{O}$ prime to the conductor factorise uniquely as product of prime ideals

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Factor the ideals $\mathfrak{b} = \mathfrak{b}_e \mathfrak{b}_k, \mathfrak{c} = \mathfrak{c}_e \mathfrak{c}_k$ with action of $\mathfrak{b}_e, \mathfrak{c}_e$ “easy”

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$$u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k) = 2^f$$

easier to solve because $\mathrm{N}(\mathfrak{b}_k)\mathrm{N}(\mathfrak{c}_k) \leq \mathrm{N}(\mathfrak{b})\mathrm{N}(\mathfrak{c})$

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"Easy" In practice

Ensure $N(\mathfrak{b}_e), N(\mathfrak{c}_e)$ are products of small primes split in $\mathcal{O}$

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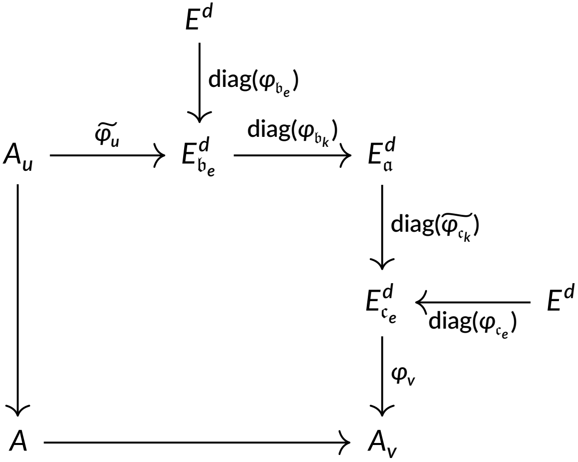
Ensure $N(\mathfrak{b}_e), N(\mathfrak{c}_e)$ are products of small primes split in $\mathcal{O}$

Concretely $\mathcal{O} = \mathbb{Z}[(1 + \sqrt{-p})/2]$

Compute $[\mathfrak{b}_e] \cdot E$ with successive Elkies isogenies defined over F_p

Solvability of the norm equation: The diagram



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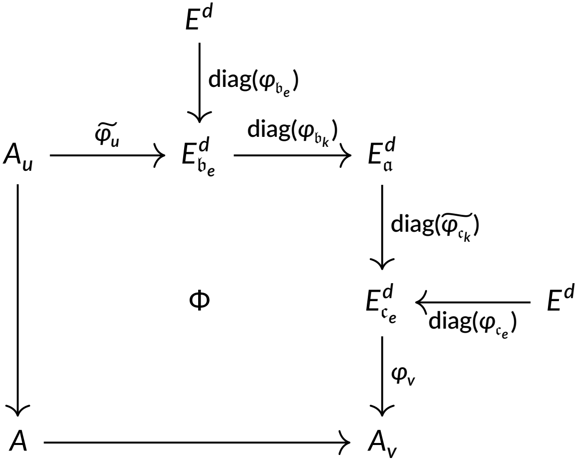

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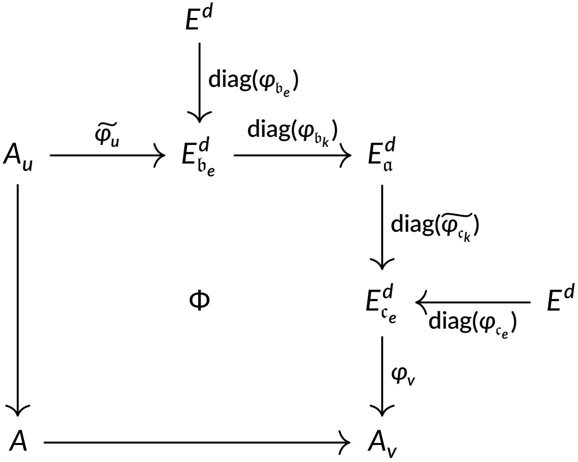

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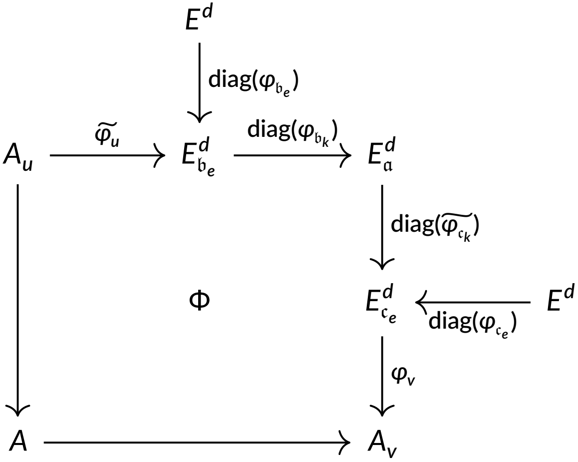

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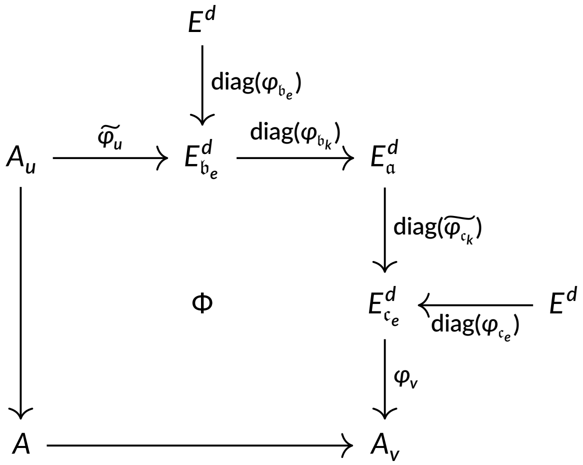
$$\ker(\Phi) = \left\{(u\mathrm{N}(\mathfrak{b}_e)x, \varphi_v \mathrm{diag}(\widetilde{\varphi_{\mathfrak{c}_k}}\varphi_{\mathfrak{b}_k})\widetilde{\varphi_u}(x)) \mid x \in A_u[2^f]\right\} \quad \text{and} \quad \widetilde{\varphi_{\mathfrak{c}_k}}\varphi_{\mathfrak{b}_k} = \frac{1}{\mathrm{N}(\mathfrak{b}_e)\mathrm{N}(\mathfrak{c}_e)}\widetilde{\varphi_{\mathfrak{c}_e}}\varphi_{\bar{\mathfrak{c}}\mathfrak{b}}\varphi_{\mathfrak{b}_e}$$

Solvability of the norm equation: The diagram



$$\varphi_{\bar{\mathfrak{c}}\mathfrak{b}} = \widetilde{\varphi_{\mathfrak{c}}}\varphi_{\mathfrak{b}} = \widetilde{\varphi_{\mathfrak{c}_e}}\varphi_{\mathfrak{c}_k}\varphi_{\mathfrak{b}_k}\widetilde{\varphi_{\mathfrak{b}_e}}$$

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Solvability of the norm equation: Some data

	Avg	Med	Min	Max
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.285	4	0	19
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.297	2	0	10
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.768	2	0	8
13-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.612	1	0	8

Table: Elkies steps required for $\log_2(p) = 33 \cdot 2^{503} - 1$.

Solvability of the norm equation: Some data

	Avg	Med	Min	Max
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.574	4	0	24
5-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	3.011	3	0	12
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.482	2	0	10
13-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.908	2	0	9

Table: Elkies steps required for $p = 15 \cdot 2^{1004} - 1$.

Solvability of the norm equation: Some data

	Avg	Med	Min	Max
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	5.582	5	0	18
5-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	3.746	4	0	11
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.402	2	0	8

Table: Elkies steps required for $p = 9 \cdot 2^{1551} - 1$.

Solvability of the norm equation: Some data

	Avg	Med	Min	Max
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.891	4	0	22
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.733	3	0	9
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.172	2	0	9
17-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.785	2	0	9

Table: Elkies steps required for $p = 51 \cdot 2^{2026} - 1$.

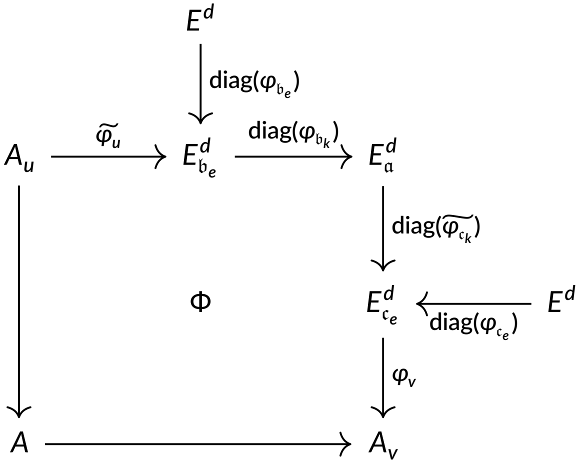
Solvability of the norm equation: Some data

	Avg	Med	Min	Max
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.950	4	0	17
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.658	2	0	9
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.010	2	0	9
17-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.691	1	0	7
19-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.567	1	0	6

Table: Elkies steps required for $p = 63 \cdot 2^{4084} - 1$.

Reminder of the diagram

$[\mathfrak{a}] = [\mathfrak{b}] = [\mathfrak{c}] \quad \mathfrak{b} = \mathfrak{b}_e \mathfrak{b}_k, \mathfrak{c} = \mathfrak{c}_e \mathfrak{c}_k$



Problem Need to construct φ_u, φ_v

Constructing isogenies of prescribed degree

The isogenies φ_u, φ_v

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Dimension 1 $\rightsquigarrow$ Dimension 2 Kani-isogeny Φ

Requires knowledge of the endomorphism ring (SQISign2D)

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Sums of squares (or QFESTA-style splitting using 4-dimensional isogenies)

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Dimension 4 $\rightsquigarrow$ Dimension 8 Kani-isogeny Φ

Zahrin's trick

Constructing isogenies of prescribed degree

In dimension 2 with sum of two squares

Constructing isogenies of prescribed degree

In dimension 2 with sum of two squares

$$u = x_u^2 + y_u^2 \qquad M_u = \begin{pmatrix} x_u & y_u \\ -y_u & x_u \end{pmatrix} : E^2 \to E^2 \qquad \deg_{\mathrm{p}}(\varphi_u) = u$$

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Theorem

$u = p_1^{k_1} \cdots p_n^{k_n} = x_u^2 + y_u^2$ if and only if k_i even when $p_i \equiv 3 \pmod 4$

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An Algorithm for constructing isogenies

of prescribed degree in dimension 2

Input: Integer u, Bound B, Set of small split primes $\mathcal{S}$

Output: Isogeny φ_u with degree u

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6. Return

$$\varphi_u = \begin{pmatrix} \psi_u & 0 \\ 0 & \psi_u \end{pmatrix} \begin{pmatrix} x_u & y_u \\ -y_u & x_u \end{pmatrix}$$

An algorithm for solving the norm equation

Input: $\mathfrak{a} \subset \mathcal{O}$, $p = c2^e - 1$, Set of small split primes $\mathfrak{B}$

Output: Return $\mathfrak{b}_e, \mathfrak{b}_k, \mathfrak{c}_e, \mathfrak{c}_k, \varphi_u, \varphi_v$ such that $u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k) = 2^f \leq 2^{e-3}$

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2. Use b_1, b_2 to iterate over small ideals $\mathrm{N}(\mathfrak{b})$ equivalent to $\mathfrak{a}$

An algorithm for solving the norm equation

Input: $\mathfrak{a} \subset \mathcal{O}$, $p = c2^e - 1$, Set of small split primes $\mathfrak{B}$

Output: Return $\mathfrak{b}_e, \mathfrak{b}_k, \mathfrak{c}_e, \mathfrak{c}_k, \varphi_u, \varphi_v$ such that $u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k) = 2^f \leq 2^{e-3}$

1. Perform lattice reduction on $\mathfrak{a}$ to obtain small basis b_1, b_2
2. Use b_1, b_2 to iterate over small ideals $\mathrm{N}(\mathfrak{b})$ equivalent to $\mathfrak{a}$
3. Factor the ideals $\mathfrak{b} = \mathfrak{b}_e\mathfrak{b}_k$ so that $\mathrm{N}(\mathfrak{b}_e)$ is a product of primes in $\mathfrak{B}$

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1. Perform lattice reduction on $\mathfrak{a}$ to obtain small basis b_1, b_2
2. Use b_1, b_2 to iterate over small ideals $\mathrm{N}(\mathfrak{b})$ equivalent to $\mathfrak{a}$
3. Factor the ideals $\mathfrak{b} = \mathfrak{b}_e\mathfrak{b}_k$ so that $\mathrm{N}(\mathfrak{b}_e)$ is a product of primes in $\mathfrak{B}$
4. Choosing pairs $\mathfrak{b} = \mathfrak{b}_e\mathfrak{b}_k, \mathfrak{c} = \mathfrak{c}_e\mathfrak{c}_k$, try to solve $u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k) = 2^f < 2^{e-3}$

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Rerandomisation

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Let's look at some data ...

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	0.149	0.111	0.043	2.286
Rerandomisations:	0.119	0	0	11
$\log(2^f = u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k))$	494.284	495	476	500
UV solutions tried:	845.562	480	0	11020
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.285	4	0	19
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.297	2	0	10
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.768	2	0	8
13-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.612	1	0	8
3-Elkies steps for φ_u, φ_v:	0.496	0	0	1
7-Elkies steps for φ_u, φ_v:	0.256	0	0	1
11-Elkies steps for φ_u, φ_v:	0.165	0	0	1
13-Elkies steps for φ_u, φ_v:	0.000	0	0	0

Table: Times, rerandomisations and elkies steps required for $\log_2(p) = 33 \cdot 2^{503} - 1$.

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	0.364	0.286	0.067	3.935
Rerandomisations:	0.061	0	0	7
$\log(2^f = u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k))$	994.723	995	974	1001
UV solutions tried:	3810.389	2305	0	43378
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.574	4	0	24
5-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	3.011	3	0	12
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.482	2	0	10
13-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.908	2	0	9
3-Elkies steps for φ_u, φ_v:	0.505	1	0	1
5-Elkies steps for φ_u, φ_v:	0.000	0	0	0
7-Elkies steps for φ_u, φ_v:	0.250	0	0	1
13-Elkies steps for φ_u, φ_v:	0.173	0	0	1

Table: Times, rerandomisations and elkies steps required for $p = 15 \cdot 2^{1004} - 1$.

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	2.396	1.268	0.135	25.267
Rerandomisations:	1.588	0	0	29
$\log(2^f = u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k))$	1544.054	1545	1531	1548
UV solutions tried:	5436.683	1991	1	87007
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	5.582	5	0	18
5-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	3.746	4	0	11
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.402	2	0	8
3-Elkies steps for φ_u, φ_v:	0.513	1	0	1
5-Elkies steps for φ_u, φ_v:	0.000	0	0	0
11-Elkies steps for φ_u, φ_v:	0.181	0	0	1

Table: Times, rerandomisations and elkies steps required for $p = 9 \cdot 2^{1551} - 1$.

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	4.148	2.181	0.163	57.346
Rerandomisations:	1.489	0	0	26
$\log(2^f = u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k))$	2018.792	2019	1998	2023
UV solutions tried:	6873.200	2816	0	72885
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.891	4	0	22
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.733	3	0	9
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.172	2	0	9
17-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.785	2	0	9
3-Elkies steps for φ_u, φ_v:	0.520	1	0	1
7-Elkies steps for φ_u, φ_v:	0.260	0	0	1
11-Elkies steps for φ_u, φ_v:	0.162	0	0	1
17-Elkies steps for φ_u, φ_v:	0.000	0	0	0

Table: Times, rerandomisations and elkies steps required for $p = 51 \cdot 2^{2026} - 1$.

Some data for the solvability of the norm equation

	Avg	Med	Min	Max
Time:	37.137	24.506	0.968	542.416
Rerandomisations:	0.499	0	0	11
$\log(2^f = u\mathrm{N}(\mathfrak{b}_k) + v\mathrm{N}(\mathfrak{c}_k))$	4076.080	4077	4058	4081
UV solutions tried:	38952.563	19701	10	548118
3-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	4.950	4	0	17
7-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.658	2	0	9
11-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	2.010	2	0	9
17-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.691	1	0	7
19-Elkies steps for $\varphi_{\mathfrak{b}_e}, \varphi_{\mathfrak{c}_e}$:	1.567	1	0	6
3-Elkies steps for φ_u, φ_v:	0.496	0	0	1
7-Elkies steps for φ_u, φ_v:	0.259	0	0	1
11-Elkies steps for φ_u, φ_v:	0.160	0	0	1
17-Elkies steps for φ_u, φ_v:	0.000	0	0	0
19-Elkies steps for φ_u, φ_v:	0.106	0	0	1

Table: Times, rerandomisations and elkies steps required for $p = 63 \cdot 2^{4084} - 1$.

Our algorithm

What we implemented

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Step 1: Finding UV

Do (easy) Lagrange lattice reduction on the input ideal $\mathfrak{a}$

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Solve the twisting problem

Timings of the steps

Parameter	Find UV	Elkies	4D	Tot. Time
500	0.097s	0.48s	0.96s	1.53s
1000	0.21s	1.16s	2.84s	4.21s
1500	1.19s	2.85s	6.49s	10.5s
2000	1.68s	8.34s	11.3s	21.3s
4000	15.6s	52.8s	53.5s	122s

Table: SageMath 10.5 timings on Intel Core i5-1235U at 4.0 GHz, in wall-clock time.

Thank you

Paper https://eprint.iacr.org/2025/401

Implementation https://github.com/pegasis4d

Slides https://rueg.re/pegasis-swissogeny

Ask me anything (I have bonus slides!)

Constructing isogenies of prescribed degree

In dimension 2 with QFESTA splitting via 4-dimensional isogeny

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Then

$$\gamma = \begin{pmatrix} \gamma_1 & \overline{\gamma_1} \\ -\gamma_2 & \overline{\gamma_2} \end{pmatrix} : E^2 \to E^2 \qquad \text{has} \qquad \deg_p(\gamma) = (x_1^2 + x_2^2) + \Delta(y_1^2 + y_2^2)$$

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Idea If $N \gg |\Delta|$ then $m = N - |\Delta|(y_1^2 + y_2^2) \geq 0$ often enough that we find $m = x_1^2 + x_2^2$ as sum of squares

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Heuristic If $N \gg |\Delta|$, we can find endomorphism of E^2 with polarised degree N

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has polarised degree $N = N_1 + N_2$ and kernel $\left\{(\deg_p(\mu_1)x, \gamma(x)) \mid x \in E^2[N]\right\}$

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Idea (Specific to Frobenius orientation)

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Idea (Specific to Frobenius orientation)

Our $u \approx \sqrt{\Delta} = \sqrt{p}$

Set $N = u(2^{e/2} - u) \gg |\Delta| = p$ to get 4-dimensional isogeny Γ of degree $2^{e/2}$

Obtain u-isogeny $\mu_i : E^2 \to A_u$ as component of Γ

Improved timings

Parameter	Find UV	Elkies	Exp. Total time	Prev. Total time	N. Rerand.
2000	0.49 s	3.83 s	21.57 s	21.3 s	0.70
4000	3.25 s	22.8 s	106.25 s	122 s	1.25

Table: Step 1 and Step 2 when solving the norm equation with single sum of squares, in wall-clock seconds.

Working over F_p: Fast basis sampling

Let $p \equiv 7 \pmod 8$ and E/F_p oriented by $\mathbb{Z}\left[\left(\sqrt{-p}+1\right)/2\right]$

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$$E[2^e] \cong {}^{\mathbb{Z}}/_{2\mathbb{Z}} + {}^{\mathbb{Z}}/_{2^{e-1}\mathbb{Z}}$$

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Let $T_{\text{desc}}, T_{\text{horiz},1}, T_{\text{horiz},2}$ points of 2-torsion

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Lemma

Let $E : y^2 = g(x)$

An element x_p in F_p lifts to $P = (x_p, y_p)$

(i) on E with $\text{ord}(P) = 2^{e-1}$ iff $x_p - x(T_{\text{desc},1})$ a non-zero non-square (and $g(x_p)$ non-zero square)

(ii) on E^t with $\text{ord}(P) = 2^{e-1}$ iff $x_p - x(T_{\text{desc},2})$ non-zero square (and $g(x_p)$ a non-zero non-square)