Workshop on the Mathematics of Post-Quantum Cryptography
6th of June 2025

On Cryptographic Group Actions from Isogenies

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Cryptographic Group Actions

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(Commutative) Cryptographic Group actions are a powerful construction

Can build many cryptographic primitives

- NIKE, (Threshold) Signatures, SSP-OT, PRF, ...

Elliptic curves

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Curves

Groups

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Base field $k = \mathbb{F}_q$

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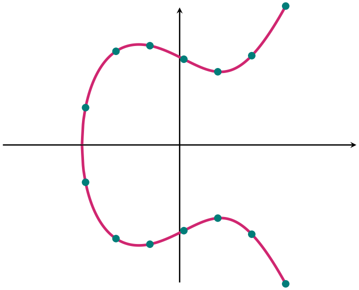

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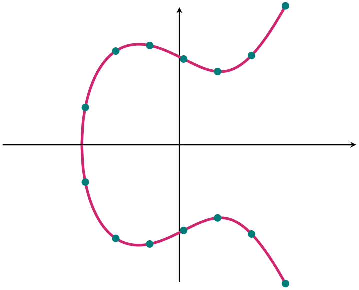

$E(L) = \{(x, y) \in E \mid x, y \in L\}$

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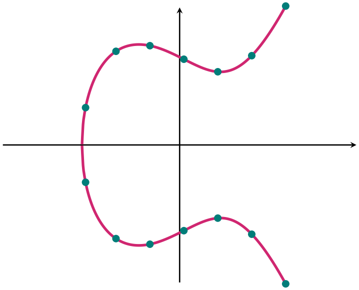

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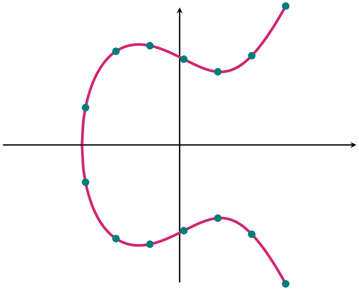

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Groups

$$P = (P_x, P_y), Q = (Q_x, Q_y) \in E$$

When $P_x \neq Q_x$

$$(P+Q)_x = \left(\frac{Q_y - P_y}{Q_x - P_x}\right)^2 - P_x - Q_x$$

$$(P+Q)_y = \left(\frac{Q_y - P_y}{Q_x - P_x}\right)(P_x - (P+Q)_x) - P_y$$

When $P_x = Q_x, P_y = Q_y \neq 0$

$$(P+Q)_x = \left(\frac{3P_x^2 + a}{2P_y}\right)^2 - 2P_x$$

$$(P+Q)_y = \left(\frac{3P_x^2 + a}{2P_y}\right)(P_x - (P+Q)_x) - P_y$$

When $P_x = Q_x, P_y = -Q_y$

$$P + Q = \mathcal{O}$$

Isogenies

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Morphism of Curves

$$\varphi : E \to E' \quad (x, y) \mapsto \left(\frac{\varphi_1(x,y)}{\varphi_2(x,y)}, \frac{\varphi_3(x,y)}{\varphi_4(x,y)} \right)$$

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First isomorphism theorem

$$\{G \subseteq E \text{ finite}\} \leftrightsquigarrow \{\text{nonzero isogenies } E \to *\}$$

$$G \mapsto \varphi_G \text{ with } \ker(\varphi_G) = G$$

$$\ker(\varphi) \leftarrow\!\shortmid \varphi$$

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Because nonzero isogenies $E \to E'$ are surjective, we write $E' = E/\ker(\varphi)$

Isogeny Problem

Given two isogenous elliptic curves, find an isogeny between them

Cryptography from Isogenies

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Common isogeny regime

$$\begin{tikzcd}
E_0 \arrow[rr, "\varphi"] & & E \\
\text{Public Parameter} \arrow[u, dashed] & \text{Secret Key} \arrow[u, dashed] & \text{Public Key} \arrow[u, dashed]
\end{tikzcd}$$

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Attempt 2 Global Atlas: Restrict to special subgroups

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Attempt 1 *Torsion point information* (SIKE, Polynomial time classical attack)

Attempt 2 *Orientations* (CRS, CSIDH, Subexponential Quantum Attack)

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Idea Alice and Bob pick secret data from $\mathbb{Z}[\pi]$

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$$A = \bigcap_{\sigma \in I_A} \ker(\sigma : E \to E) = \ker(\alpha_1 : E \to E) \cap \ker(\alpha_1 : E \to E)$$

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Conclusion

We can turn ideals $I_A \subseteq \mathbb{Z}[\pi]$ into isogenies $\varphi_A : E \to E_A$ starting on any curve E

...in particular we can use I_A to go from E to E_A by computing φ_A

The Global Atlas: The Class Group

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Fact I, J ideals, then $E \xrightarrow{\varphi_I} E_I \xrightarrow{\varphi_J} E_{IJ}$ is the same as $E \xrightarrow{\varphi_{IJ}} E_{IJ}$

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- $I \subseteq \mathbb{Z}[\pi]$ ideal then $I\bar{I} = \mathrm{N}(I)\mathbb{Z}[\pi]$ principal
- Q is a multiplicative group, where $\bar{I}$ is the inverse of I
- Q is the Ideal Class Group $\mathrm{Cl}(\mathbb{Z}[\pi])$

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Theorem

$\mathrm{Cl}(\mathbb{Z}[\pi])$ acts regularly on all supersingular elliptic curves defined over $\mathbb{F}_p$

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If $\mathrm{N}(\mathcal{I}) = \prod p_i^{e_i}$ the cost of evaluation is $\widetilde{\Theta}\left(\sum e_i p_i\right)$

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...finds J with $\sqrt{p} \approx \mathrm{N}(J) \leq \mathrm{N}(I)$

...this does not help with smoothness

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Only act by ideals with smooth norm!

These can be decomposed as the action of many small-normed ideals

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Result Practically and asymptotically efficient restricted action

PEGASIS: Unrestricting using Higher-Dimensions

PEGASIS: Practical Effective Class Group Action using 4-Dimensional Isogenies

Joint with Pierrick Dartois, Jonathan Komada Eriksen, Tako Boris Fouotsa, Arthur Herledan Le Merdy, Riccardo Invernizzi, Damien Robert, Frederik Vercauteren and Benjamin Wesolowski

https://eprint.iacr.org/2025/401 (To appear at Crypto'25)

First post-quantum commutative unrestricted cryptographic group action that is both asymptotically and practically efficient

Main ingredient: Clapoti

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Theorem (Page-Robert, 2023)

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Given

$[I] = [J]$ equivalent ideals, $\quad u = u_1^2 + u_2^2 + \cdots + u_g^2, \quad v = v_1^2 + v_2^2 + \cdots + v_g^2$

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Let

$$d = u\mathrm{N}(\mathcal{I}) + v\mathrm{N}(\mathcal{J}) = \prod p_i^{e_i} \quad \text{and} \quad k \text{ minimal so } E[d] \subseteq E(\mathbb{F}_{p^k})$$

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$$\tilde{\Theta}\left(k \sum e_i p_i^{2g}\right)$$

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In practice Want $d = 2^n, k = 2, g \leq 2$

Conflicting requirements

$d = 2^n$ (Target solution of $u\mathrm{N}(I) + v\mathrm{N}(J) = d$)

Guaranteed solutions for $u\mathrm{N}(I) + v\mathrm{N}(J) = 2^n$ when $2^n \geq \mathrm{N}(I)\mathrm{N}(J)$ (Coin Problem)

...but Minkowski's bound only gives us $[J] = [I]$ with $\mathrm{N}(J) \approx \sqrt{p}$

...so $p \leq 2^n$

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$k = 2$ (Extension degree we want for $E[d] \subseteq E(\mathbb{F}_{p^k})$)

E supersingular, so $\#E(\mathbb{F}_{p^2}) = (p+1)^2$ (Need supersingularity!)

...if $E[2^n] \subseteq E(\mathbb{F}_{p^2})$, then 2^n must divide $p + 1$ (because $\#E[2^n] = 2^{2n}$)

...in particular $2^n = 2^{\lfloor \log_2(p) \rfloor - \varepsilon} < p$

Conflicting requirements

$d = 2^n$ (Target solution of $u\mathrm{N}(I) + v\mathrm{N}(J) = d$)

Guaranteed solutions for $u\mathrm{N}(I) + v\mathrm{N}(J) = 2^n$ when $2^n \geq \mathrm{N}(I)\mathrm{N}(J)$ (Coin Problem)

...but Minkowski's bound only gives us $[J] = [I]$ with $\mathrm{N}(J) \approx \sqrt{p}$

...so $p \leq 2^n = 2^{\lceil \log_2(p) \rceil}$

$k = 2$ (Extension degree we want for $E[d] \subseteq E(\mathbb{F}_{p^k})$)

E supersingular, so $\#E(\mathbb{F}_{p^2}) = (p+1)^2$ (Need supersingularity!)

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...in particular $2^n = 2^{\lfloor \log_2(p) \rfloor - \varepsilon} < p$

Core Problem

Even the smallest equivalent ideals are too big

...but not by much

Resolving the tension

Resolving the tension

Idea

Resolving the tension

Idea

Factor ideal into smooth-normed and rough-normed part $I = I_s I_r, J = J_s J_r$

The smooth parts can be computed like in CSIDH

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...but they almost are $[\overline{I_r} J_r] = [I_s \overline{J_s}]$ (not trivial, but easy to evaluate)

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Tweak Page-Robert's algorithm

Search for $u, v \geq 0$ such that $d = u\mathrm{N}(I_r) + v\mathrm{N}(J_r) = 2^n$

...works because $\mathrm{N}(I_r), \mathrm{N}(J_r)$ is small enough

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In fact So many solutions u, v that we can find $u = u_1^2 + u_2^2, v = v_1^2 + v_2^2$

...algorithm only needs $(2g = 4)$-dimensional isogeny computation

In fact Using x-only arithmetic, we get away with $k = 1$

Implementation Results

		Lang.	128	256	375	512	1024
Restricted	CSIDH*	C	40ms				
	SQALE*	C					5.75s**
	dCTIDH*	C				350ms**	
Unrestricted	SCALLOP*	C++	35s	750s			
	SCALLOP-HD*	Sage	88s	1140s			
	PEARL-SCALLOP*	C++	30s	58s	710s		
	KLaPoTi	Sage	207s				
		Rust	1.95s				
	PEGASIS	Sage	1.53s	4.21s	10.5s	21.3s	121s

Table: *Measured on different hardware, **Converted from cycles to time @4GHz.

Thank you for your attention

PEGASIS Paper https://eprint.iacr.org/2025/401 (To appear at Crypto'25)

PEGASIS Implementation https://github.com/pegasis4d

Slides https://rueg.re/mathsofpqc25

Ask me anything (I have bonus slides!)

The Ideal2Isogeny Construction

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Want to compute $[\mathfrak{a}] \cdot E = E_{\mathfrak{a}}$

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& E
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$$\begin{tikzcd}
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\end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{aligned}
\ker(\Phi) &= \left\{ \left(\mathrm{N}(\mathfrak{b})x, \widetilde{\varphi_{\mathfrak{c}}}\varphi_{\mathfrak{b}}(x) \right) \mid x \in E[\deg_{\mathrm{p}}(\Phi)] \right\} \\
&= \left\{ \left(\mathrm{N}(\mathfrak{b})x, \varphi_{\bar{\mathfrak{c}}\mathfrak{b}}(x) \right) \mid x \in E[\deg_{\mathrm{p}}(\Phi)] \right\}
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\end{aligned}$$

Norm equation $\deg_\mathrm{p}(\Phi) = \mathrm{N}(\mathfrak{b}) + \mathrm{N}(\mathfrak{c}) \stackrel{!}{=} 2^f$

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$$\begin{aligned}
\ker(\Phi) &= \left\{ \left(\mathrm{N}(\mathfrak{b})x, \widetilde{\varphi_{\mathfrak{c}}}\varphi_{\mathfrak{b}}(x) \right) \mid x \in E[\deg_p(\Phi)] \right\} \\
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Requirements

1. $f \le v_2(p+1) - 3$
2. $\mathrm{N}(\mathfrak{b}), \mathrm{N}(\mathfrak{c})$ coprime

The Ideal2Isogeny Construction: Iteration #2

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Let $[\mathfrak{a}] = [\mathfrak{b}] = [\mathfrak{c}]$, $\varphi_u : E \to E_u$, $\varphi_v : E \to E_v$

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$$\begin{aligned}
&\ker(\Phi) \\
&= \left\{ \left(u\mathrm{N}(\mathfrak{b})x, \varphi_v \widetilde{\varphi}_{\mathfrak{c}} \varphi_{\mathfrak{b}} \widetilde{\varphi}_u(x) \right) \mid x \in E_u[\deg_{\mathrm{p}}(\Phi)] \right\} \\
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Norm equation $\deg_{\mathrm{p}}(\Phi) = u\mathrm{N}(\mathfrak{b}) + v\mathrm{N}(\mathfrak{c}) \overset{!}{=} 2^f$

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Want to compute $[\mathfrak{a}] \cdot E = E_{\mathfrak{a}}$

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Let $[\mathfrak{a}] = [\mathfrak{b}] = [\mathfrak{c}]$, $\varphi_u : E^d \to A_u$, $\varphi_v : E^d \to A_v$

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& & E^d \arrow[d, "\varphi_v"] \\
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Working over F_p: Fast basis sampling

Let $p \equiv 7 \pmod 8$ and E/F_p oriented by $\mathbb{Z}\left[\left(\sqrt{-p}+1\right)/2\right]$

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$$E[2^e](F_p) \cong {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} + {}^{\mathbb{Z}}\!/_{2^{e-1}\mathbb{Z}}$$

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Let $T_{\text{desc}}, T_{\text{horiz},1}, T_{\text{horiz},2}$ points of 2-torsion

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Let $T_{\text{desc}}, T_{\text{horiz},1}, T_{\text{horiz},2}$ points of 2-torsion

Lemma

Let $E : y^2 = g(x)$

An element x_p in F_p lifts to $P = (x_p, y_p)$

(i) on E with $\operatorname{ord}(P) = 2^{e-1}$ iff $x_p - x(T_{\text{desc},1})$ a non-zero non-square (and $g(x_p)$ non-zero square)

(ii) on E^t with $\operatorname{ord}(P) = 2^{e-1}$ iff $x_p - x(T_{\text{desc},2})$ non-zero square (and $g(x_p)$ a non-zero non-square)