Leuven Isogeny Days 6, COSIC, 12th of September 2025

A presentation on work in progress

On the Concrete Classical Hardness of the Supersingular Isogeny Problem

Joint with Lorenz Panny and Alessandro Sferlazza

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Concreteness and measuring the cost of attacks

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    - cost of memory access is $O(\sqrt{M})$ (McEliece and NTRU explicitly mention this)

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3. Hardware implementations of specific subroutines
   - VLSI model with Area-Time cost measure
   - Can design ASICs or FPGAs (simple chips)
   - Performance can be evaluated through simulation
   - Used in analysis of SIKE and lead to suggested lower parameters

     [Longa-Wang-Szefer20]

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4. Full best-effort implementation on powerful hardware
    - Allows one to test assumptions about e.g. memory constraints
    - Gives real-world numbers
    - In good cases, breaking small instances can be extrapolated to big instances
    - It can be used constructively for non-cryptographic sizes!
    - Example “Jessica” g6k lattice sieving tools

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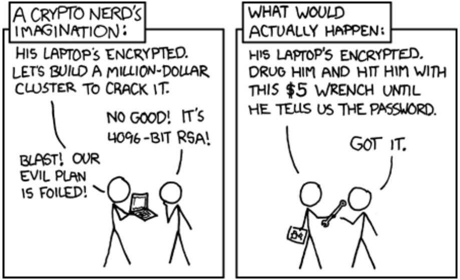

https://xkcd.com/538

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This work

- We implemented a GPU accelerated isogeny-graph explorer, that can navigate to the $\mathbb{F}_p$ subgraph from a random starting curve in a few hours working over [redacted]-bit generic primes
- We also have machinery to perform vectorisation over $\mathbb{F}_p$
- ...and from this, to recover the full endomorphism ring in practically efficient time

Supersingular isogeny problems

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Given two supersingular elliptic curves E_1, E_2 find an ℓ^k-isogeny $E_1 \to E_2$

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They are equivalent ... but how would one solve them *in practice*?

Attacks against supersingular isogeny problems

Meet in the Middle (for isogeny path problem)

Idea

Between two random curves E_1, E_2 there exists an ℓ^k isogeny with $k = O(\log_\ell(p))$

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Can be improved with van Oorschot-Wiener Golden Collision Search

...to give cost $\tilde{O}(p^{3/4}/M^{1/2}/C)$ time and $\tilde{O}(M)$ memory using C cores

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Do random ℓ-isogeny walks $E_1 \to \cdots \to E_1', E_2 \to \cdots \to E_2'$ until E_1', E_2' are over $\mathbb{F}_p$

Find isogeny over $\mathbb{F}_p$ between $E_1' \to E_2'$ (e.g. with vOW Golden Collision search)

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Concretely

First stage is naturally memoryless. Memory for second stage can be configured

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However knowing an orientation is tantamount to knowing an endomorphism

- If you can (easily) find endomorphisms of a given curve, you can solve the isogeny problem already (OneEnd ⇔ EndRing ⇔ IsogenyPathProblem)

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Concretely Memoryless, but requires multiple calls to first step (and larger ℓ_j)

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This is really the asymptotic bottleneck

...so how do we do this, for real?

Concrete Questions and Improvements

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  ...so $O(\ell^2)$ $\mathbb{F}_p$-multiplications per node visited
  ...so choose $\ell = 2$

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Better detection i.e. SuperSolver [Corte-Real Santos-Costello-Shi21]

- Core Insight one step costs half a square root
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  ...which costs $O(\log(p))$ $\mathbb{F}_p$-multiplications
- Therefore any test for orientability that uses constant number of $\mathbb{F}_p$ multiplications will eventually become more efficient as p grows

NeighbourIsOriented

NeighbourInFp + Generalised Delfs-Galbraith

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Naively

checking whether ℓ-neighbours of E are d-isogenous to their $\mathbb{F}_p$-conjugate

NeighbourIsOriented

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checking whether ℓ-neighbours of E are d-isogenous to their $\mathbb{F}_p$-conjugate

- Find neighbours: compute solutions of $\Phi_\ell(j(E), z) = 0 \pmod p$
- Is oriented: verify whether any solution z satisfies $\Phi_d(z, z^p) = 0 \pmod p$

NeighbourIsOriented

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Idea Avoid root computations

$$
\begin{aligned}
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Finally check whether $r_0(y) = r_1(y) = 0$ has solution by computing $\gcd(r_0(y), r_1(y))$

Generalised NeighbourInFp: NeighbourIsOriented

Indeed this generalises NeighbourInFp

1. $\ell = 1, d = 1$ Checks whether E is in $\mathbb{F}_p$
2. $\ell > 1, d = 1$ Checks whether E has ℓ-neighbours in $\mathbb{F}_p$ (SuperSolver)
3. $\ell = 1, d > 1$ Checks whether E is $\mathbb{Z}[\sqrt{-dp}]$-oriented (Generalised Delfs-Galbraith)
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There is interesting concurrent work that might improve on this

Minimising costs of checking NeighbourIsOriented

Given a set of tests $T_{\ell,c}$, we must minimise

$$\min_{I}\left(\frac{\mathrm{cost}(\sqrt{p})/2 + \sum_{(\ell,c)\in I} \mathrm{cost}(T_{\ell,c})}{\sum_{(\ell,c)\in I} \mathrm{p.success}(T_{\ell,c})}\right)$$

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Example Supersolver did this for
$c = 1, \ell = 3, 5, 7, 9, 11, 13, 17, 19, 15, 23, 25, 29, 21, 31, 27, 37, 41, 43, 33, 35$

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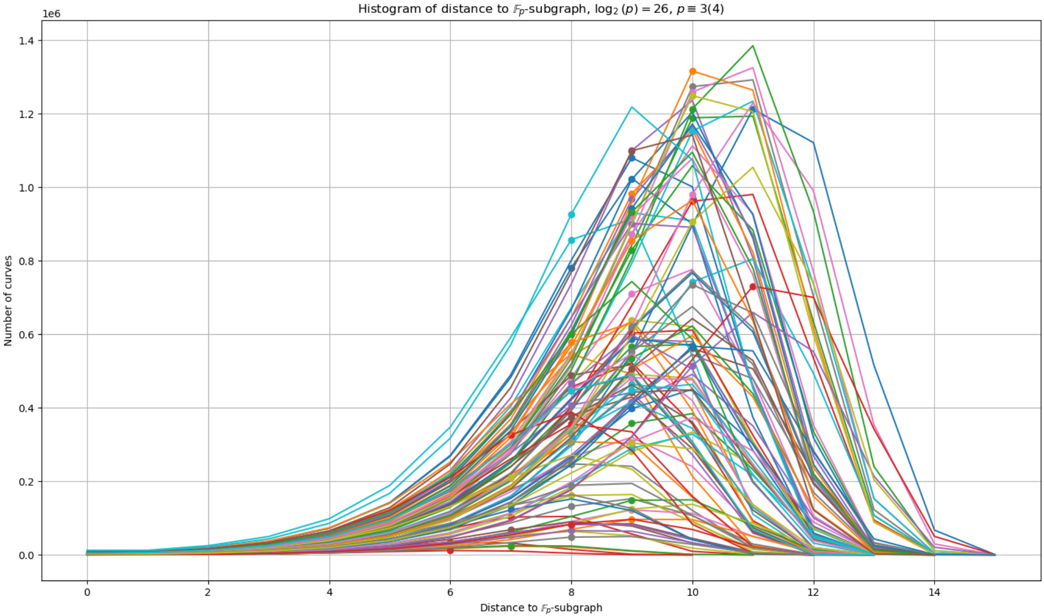
We are trying to get better estimates for both asymptotic and concrete costs (Work in progress!)

Bad Neighbourhoods and Rerandomisation: Dandelions



Why GPUs

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Example NVIDIA L40s

- 19,000 cuda cores
- 32 bit architecture
- Streaming multi-processors each manage 128 cuda cores
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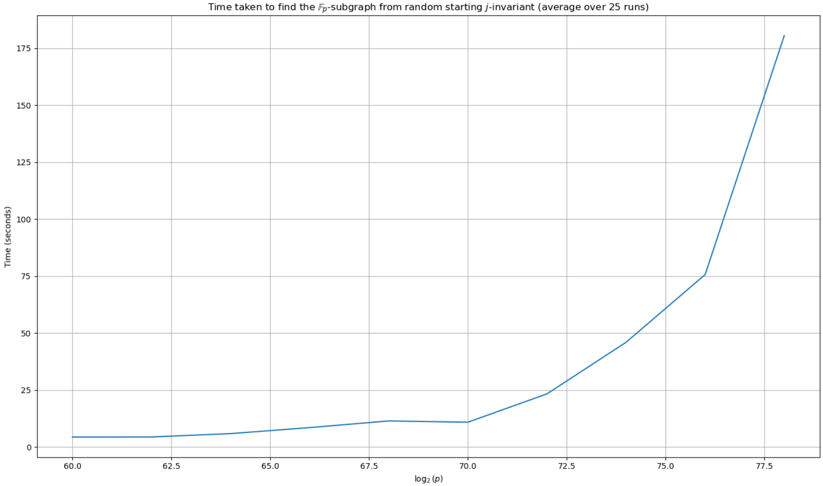
GPUs aren't just fast

- Specialised hardware: NSA will do similar things
- Similar problems (limited memory, simple logic, not superscalar)
- Close(r) to silicon

Some timings

Very Preliminary timings (WIP!)



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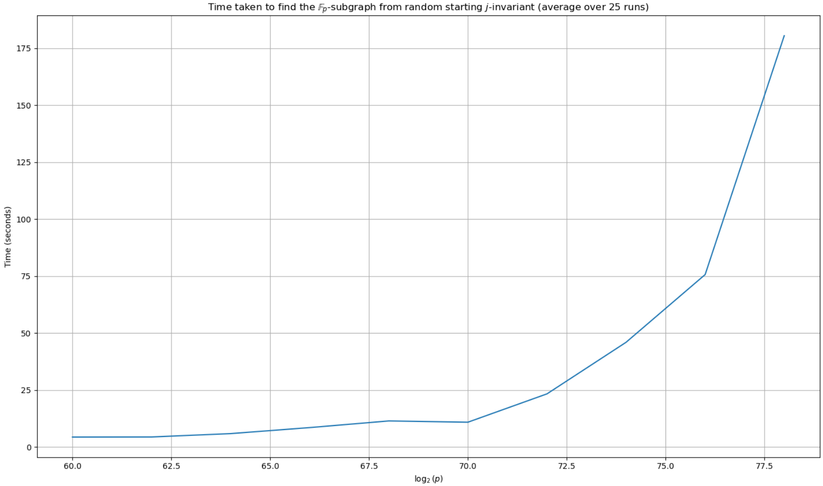

Larger instances We can break 95bit instances in just a few hours on average

Many Questions and Todos

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Clustering of the $\mathbb{F}_p$ subgraph (Adventures in Supersingularland)

Can we efficiently detect oriented neighbours?

Use of radical isogenies and torsion to walk through the graph [Chi-Domínguez25]

Summary

Practical

We wrote a GPU accelerated implementation of the SuperSolver variant of the Delfs-Galbraith attack to break instances of the supersingular isogeny problem for 95 bit generic primes in a few hours

This attack would cost ~30 USD on AWS and ~5 USD in the Hetztner cloud

Theoretical

We combined the theory of SuperSolver's NeighbourInFp detection with Generalised Delfs-Galbraith to obtain a NeighbourIsOriented detection subroutine, but we think that this turns out to be an unfavourable generalisation

Thank you for your attention

Slides https://rueg.re/lid25

RFC: SQISign Challenges

Many schemes have public challenges that give prize money for breaking their non-cryptographic parameters (e.g. RSA, SIKE)

Perhaps we want to do this for SQISign? Or give a general isogeny challenge?

An apparent obstruction is the trusted setup required to generate these challenges and then a zero-knowledge protocol to prove that a solution has been found

Hopefully we will soon have a better idea of how expensive attacks are in practice, and can start thinking about setting challenges