Computing the Isogeny Class-Group Action on Ordinary Elliptic Curves by going into higher dimensions

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Outline

Cryptographic Group Actions

Motivation Diffie-Hellman key exchange is not post-quantum

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Closer Look

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Rephrase NIKE from group action

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Note These are equivalently hard problems on a quantum computer [GPSV18, MZ22, GLM24]

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+ more (Smooth & Projective Hashing, Dual-Mode PKE, SSP-OT, Naor-Reingold PRF) [ADFMP20]

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Naturally arising CGA from isogenies: *Isogeny Class-Group Action* [Cou97, RS06]

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Classical [GHS01]

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Quantum [Kup10, CJS10]

Complexity $O\left(\exp\left(\sqrt{\log(|H|)}\right)\right)$

Elliptic curves and Isogenies

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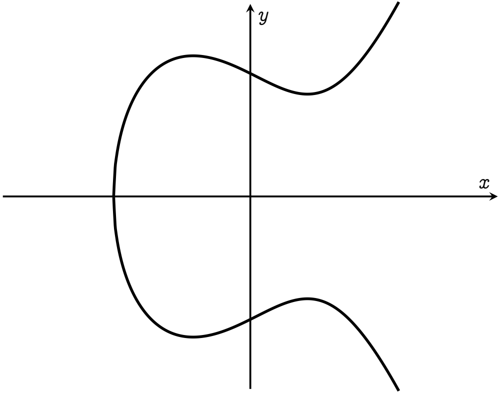
Algebraic

$$y^2 = x^3 + ax + b$$



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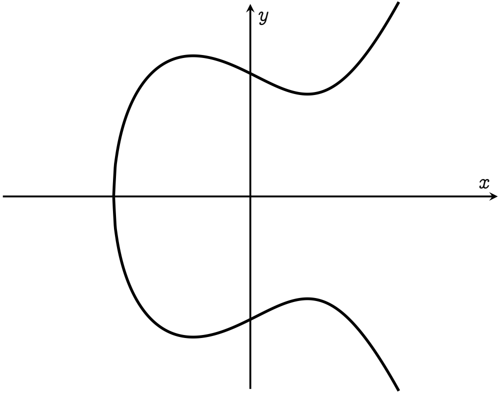

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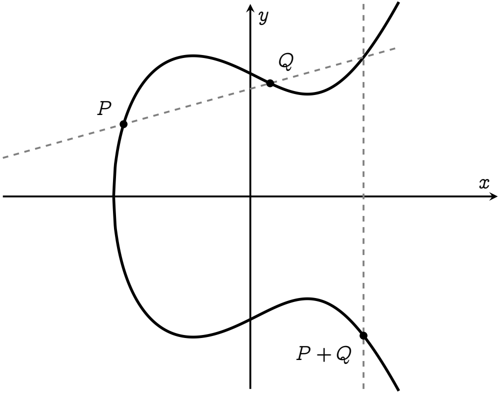

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Example Scalar multiplication

$$[N]_E \colon E \to E \qquad P \mapsto NP = \underbrace{P + \cdots + P}_{N \text{ times}}$$

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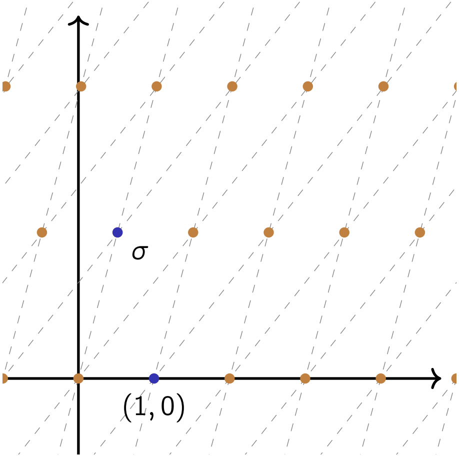

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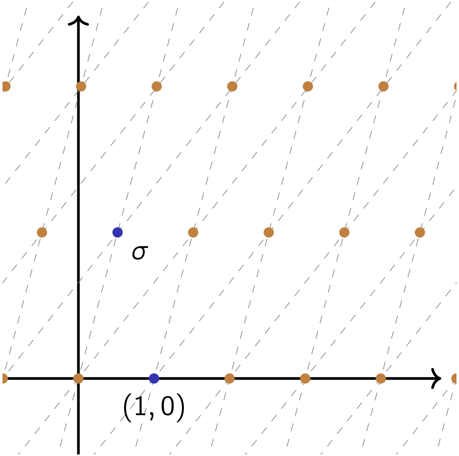
Lattice + Ring = *Order* $\rightsquigarrow \mathrm{End}(E) \cong \mathcal{O}$



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Computing this isogeny class-group action

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Input: $[I] \in \mathrm{Cl}(\mathcal{O})$ and $[E] \in \mathrm{Ell}(\mathcal{O})$

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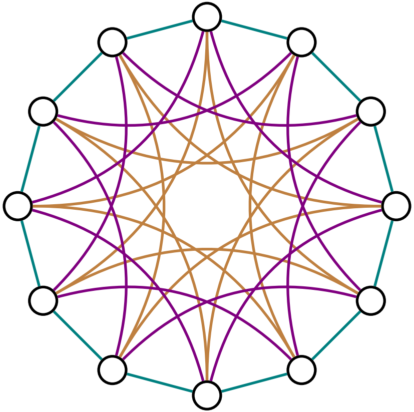
Problem $J \colon E \to E_J$ might have large non-smooth degree

Computing this isogeny class-group action



Higher dimensional methods: Abelian Varieties

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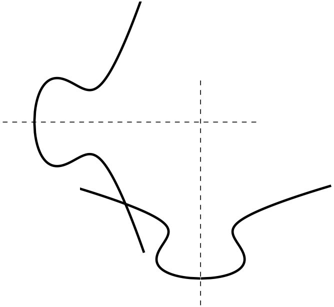
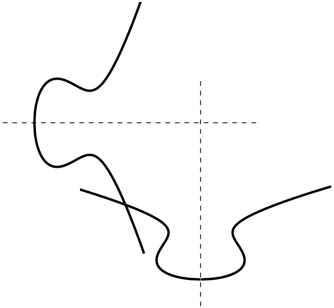


Higher dimensional methods: Abelian Varieties



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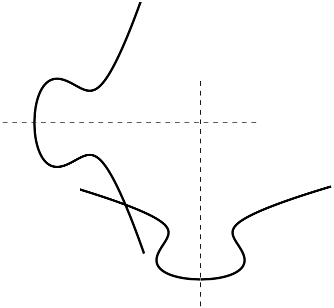

Resolution

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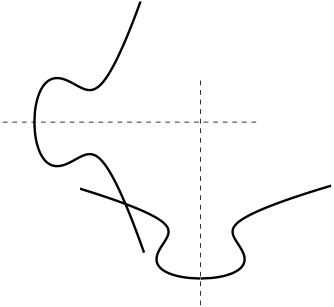

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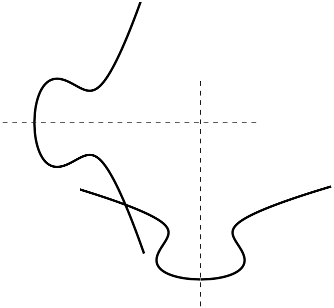

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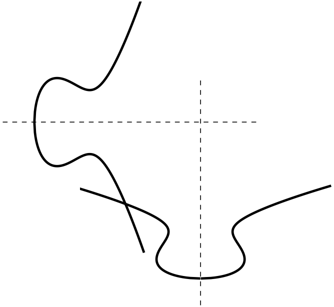

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Example Scalar multiplication $[N]_A \colon A \to A; P \mapsto NP = P + \cdots + P$

Abelian Varieties: Features of isogenies

Lemma

For all isogenies

$$\varphi\colon E_1 \times \cdots \times E_n \to E_1' \times \cdots \times E_n'$$

the reverse map exists

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Naïve Evaluating an isogeny of degree d costs $O(d)$

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E_1 \times \cdots \times E_n \arrow[rrrr, "\varphi"] \arrow[dr, bend right, "\varphi_1"'] & & & & E'_1 \times \cdots \times E'_n \\
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Evaluating φ_i requires heavy machinery: *Mumford's Theta Coordinates*

Going into higher dimensions: Kani's Lemma

Lemma (Kani)

From a commuting square of isogenies between elliptic curves

$$\begin{tikzcd} E_1 \arrow[r, "g_1"] \arrow[d, "h_1"'] & E_3 \arrow[d, "g_2"] \\ E_4 \arrow[r, "h_2"'] & E_2 \end{tikzcd}$$

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$$K = \begin{pmatrix} g_1 & \widetilde{g_2} \\ -h_1 & \widetilde{h_2} \end{pmatrix} : E_1 \times E_2 \to E_3 \times E_4$$

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From a commuting square of isogenies between general Abelian varieties

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$$\begin{pmatrix} g_1 & \widetilde{g_2} \\ -h_1 & \widetilde{h_2} \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix} = \begin{pmatrix} g_1(P) \\ -h_1(P) \end{pmatrix} \quad \rightsquigarrow \quad g_1(P)$$

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$$\rightsquigarrow$$

$$\begin{tikzcd} E \arrow[r, "I"] \arrow[d, "J"'] & E_I \arrow[d, "J"] \\ E_J \arrow[r, "I"'] & E' \cong E_{IJ} \end{tikzcd}$$

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(i) Square commutes

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{[E]} \arrow[r, "{[I]}"] \arrow[dd, "{[\bar{J}]}"'] & {[E_I]} \arrow[dd, "{[\bar{J}]}"] & & E \arrow[r, "I"] \arrow[dd, "\bar{J}"'] & E_I \arrow[dd, "\bar{J}"] \\
& & \rightsquigarrow & & \\
{[E_{\bar{J}}]} \arrow[r, "{[I]}"'] & {[E]} & & E_{\bar{J}} \arrow[r, "I"'] & E \cong E_{I\bar{J}}
\end{tikzcd}$$

(i) Square commutes

(ii) Horizontal degree $\deg(I)$

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& & {} \arrow[r, rightsquigarrow] & {} & & \\
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Kani-map K has degree $N = \deg(I) + \deg(J)$

Clapoti [PR23]

Problem $N = \deg(I) + \deg(J)$ might not be smooth

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Idea

$\alpha, \beta \in \mathcal{O} \cong \mathrm{End}(E)$

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E \arrow[r, "\alpha"] \arrow[d, "\beta"'] & E \arrow[r, "I"] & E_I \arrow[d, "\beta"] \\
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Induced Kani-map degree $N = \deg(\alpha I) + \deg(\beta J) = \deg(\alpha)\deg(I) + \deg(\beta)\deg(J)$

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Slogan *"Composing with endomorphisms gives us wiggle room"*

Clapoti [PR23]

Extend this idea

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Idea Endomorphism $E^2 \longrightarrow E^2$

$$\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} : E^2 \longrightarrow E^2 \qquad\qquad a_1, a_2 \in \mathbb{Z}$$

has

$$\deg(\alpha) = a_1^2 + a_2^2$$

Clapoti [PR23]

Extend this idea

Idea Endomorphism $E^4 \to E^4$

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_4 & a_3 & -a_2 & a_1 \end{pmatrix} : E^4 \to E^4 \qquad a_1, a_2, a_3, a_4 \in \mathbb{Z}$$

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Theorem (Jacobi) Every positive integer is the sum of four squares

Clapoti [PR23]

Conclusion For any smooth N and ideals I, J find $\alpha, \beta \in \mathrm{Mat}_{4\times 4}(\mathbb{Z})$ so that

$$N = \deg(\alpha)\deg(I) + \deg(\beta)\deg(J)$$

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Then get diagram

$$\begin{tikzcd}
E^4 \arrow[r, "\alpha"] \arrow[d, swap, "\beta"] & E^4 \arrow[r, "I^{(4)}"] & E_I^4 \arrow[d, "\beta"] \\
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E_{\bar{J}}^4 \arrow[r, swap, "\alpha"] & E_{\bar{J}}^4 \arrow[r, swap, "I^{(4)}"] & E^4
\end{tikzcd}$$

where

$$I^{(4)} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : E^4 \to E_I^4$$

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Again a Kani-square!

Induced Kani-map has degree $N = \deg(\alpha)\deg(I) + \deg(\beta)\deg(J)$ smooth!

Clapoti Summary

Input: Ideal class $[H] \in \mathrm{Cl}(\mathcal{O})$, Curve class $[E] \in \mathrm{Ell}(\mathcal{O})$

Output: $[E_H]$

1. Find $I, J \subseteq \mathcal{O}$ such that: (i) $[I] = [J] = [H]$

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Fact Polynomial time in $|\mathrm{Cl}(\mathcal{O})|$, no pre-computation!

Comparison of dimensions

Method		Dimension
Direct evaluation	Only if $\deg(I)$ is smooth	1
Use endomorphisms of E	$K = E \times E \to E_I \times E_{\overline{J}}$	2
Use endomorphisms of E^2	$K = E^2 \times E^2 \to E_I^2 \times E_{\overline{J}}^2$	4
Use endomorphisms of E^4	$K = E^4 \times E^4 \to E_I^4 \times E_{\overline{J}}^4$	8

$$\begin{tikzcd} E \arrow[r, "I"] & E_I \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "\alpha I"] \arrow[d, "\beta J"'] & E_I \arrow[d, "\beta J"] \\ E_{\overline{J}} \arrow[r, "\alpha I"'] & E \end{tikzcd}$$

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Computing higher-dimensional isogenies in practice

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Recall Factoring isogenies

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[rrrr, "\varphi"] \arrow[dr, bend right, "\varphi_1"'] & & & & E'_1 \times \cdots \times E'_n \\
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Mathematically Computing φ_i requires time exponential in n

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Practically Right now, only have implementation of

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due to [DMPR23]

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Caveat This 2d-library requires very special group structure on E

Ordinary curves with special group structure

Folklore *CM Method*

Ordinary curves with special group structure

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Naïve Sage implementation

$\mathbb{Z}/2^{512}\mathbb{Z} \times \mathbb{Z}/2^{512}\mathbb{Z} \subseteq E(F_q)$ in $\sim$ 30 minutes, $\left|E(F_q)\right| \sim$ 1024 bits

$\mathbb{Z}/2^{1024}\mathbb{Z} \times \mathbb{Z}/2^{1024}\mathbb{Z} \subseteq E(F_q)$ in $\sim$ 28 hours, $\left|E(F_q)\right| \sim$ 2048 bits

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Only works for small class-groups

Best method [Sut12]

$$O\left(\left|\sqrt{\mathrm{Cl}(\mathcal{O})}\right| \log\left(\left|\sqrt{\mathrm{Cl}(\mathcal{O})}\right|\right)\right)$$

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Does not scale

Ordinary curves with special group structure: Larger Class Group

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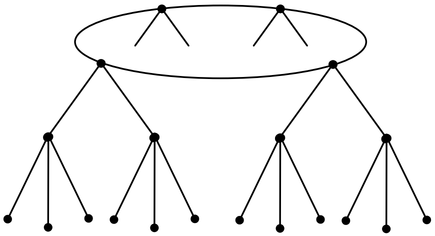
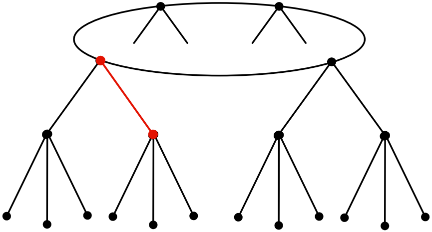
Idea

Walk down the ℓ-isogeny volcano



Ordinary curves with special group structure: Larger Class Group



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Start: $\mathrm{End}(E) \cong \mathcal{O}$

After n ℓ-steps: $\mathrm{End}(E_{\ell^n}) \cong \mathcal{O}_{\ell^n} \subseteq \mathcal{O}$

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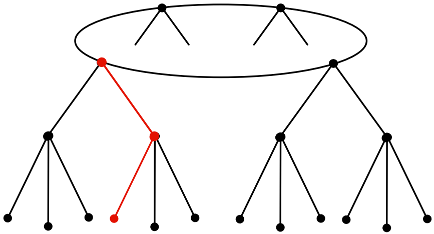

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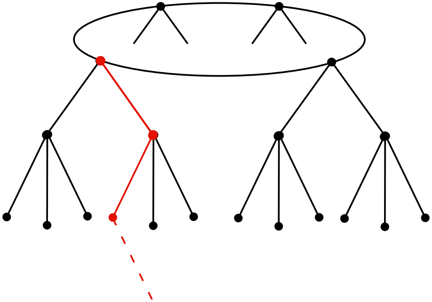

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After n ℓ-steps: $\mathrm{End}(E_{\ell^n}) \cong \mathcal{O}_{\ell^n} \subseteq \mathcal{O}$

Ordinary curves with special group structure: Larger Class Group

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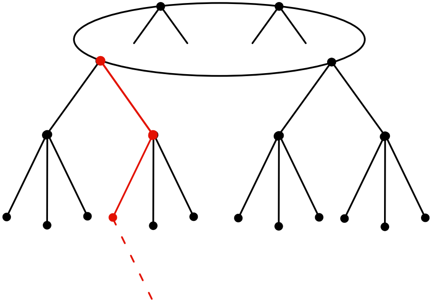

Lemma

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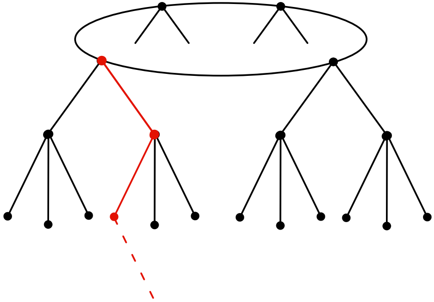

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Note This is not secure! [DDF21]

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Find ordinary E using CM Method and volcano walking

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$$N = \deg(\alpha)\deg(I) + \deg(\beta)\deg(J) = 2^n$$

2. Compute kernel of Kani-map K. Depends on $\alpha, \beta, I\overline{J}$
3. Pass $\ker(K)$ to the 2d-library [DMPR23] to obtain an equation for E_I

Solving degree equations

1.(iii) Finding endomorphisms $\alpha, \beta \in \mathcal{O}$ so that

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Case $(y \neq 0)$ numbers represented by $\deg(\alpha)$ explode with $|\mathrm{Cl}(\mathcal{O})|$

Calling the 2-dimensional isogeny library

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[rrrr, "\varphi"] \arrow[dr, bend right, "\varphi_1"'] & & & & E'_1 \times \cdots \times E'_n \\
& E''_1 \times \cdots E''_n \arrow[r, "\varphi_2"'] & \cdots \arrow[r, "\varphi_{m-1}"'] & A_{m-1} \arrow[ur, bend right, "\varphi_m"'] &
\end{tikzcd}$$

Conclusion

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Implementations

(i) CM Method

(ii) Ideals of endomorphism rings + rudimentary class group computation $\sim 2^{13}$

(iii) Finding matching endomorphisms $\sim 2^{10}$

(iv) Translation from endomorphisms to isogenies

(v) Computing the Kani-kernel

Outlook

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Curve finding

Difficult problem

Removing the requirement of special group structure

Pair finding

More precise lattice sampling techniques (α in $\mathbb{Z} + \sigma\mathbb{Z}$)

Better 2-dimensional heuristics a la SQISign-2D-West [BDDF$^+$24]

Higher-dimensional isogenies

Higher higher-dimensional isogeny-libraries

Thank you!

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